JOURNAL OF COMBINATORIAL THEORY 8, 23–29 (1970)

Connectivity of Transitive Graphs*

MARK E. WATKINS

University of Waterloo, Waterloo, Ontario, Canada

Received June 25, 1968

ABSTRACT

The conditions imposed by edge-transitivity and vertex-transitivity on the connectivity of simple graphs are investigated. Particular attention is given to the structure of those vertex-transitive graphs for which the degree of regularity exceeds the connectivity.

1. INTRODUCTION

Only finite simple graphs are considered in this note and the symbol G will always denote such an object. If G is vertex-transitive, then it is certainly regular. When the degree of regularity and the connectivity are denoted, respectively, by $\rho(G)$ and $\kappa(G)$, then

$$\rho(G) \geqslant \kappa(G). \tag{1-1}$$

For vertex-transitive graphs, we shall describe the structure of G when strict inequality holds in (1-1). We shall also obtain a best limit for the range of values of $\rho(G)/\kappa(G)$.

If G is edge-transitive, then G need not be regular. However, if G is connected, one can show that $\kappa(G)$ equals the minimum valence.

The author gratefully acknowledges some conversations with Professors C. St. J. A. Nash-Williams and Donald Miller which have helped to realize this paper.

2. PRELIMINARIES

The terminology in this note (including that of the foregoing section) is essentially that of W. T. Tutte [3]. We include, however, a few definitions for the sake of completeness.

* Presented at the Yale University Conference on Combinatorial Theory in honor of Professor Oystein Ore (May, 1968).

The group of graph-automorphisms (or symmetries) of G will be denoted by $\Gamma(G)$. $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, of G.

G is *vertex-transitive* when, for given $x, y \in V(G)$, there exists $\phi \in \Gamma(G)$ such that $\phi(x) = y$. G is *edge-transitive* when, for given $[x, y], [u, v] \in E(G)$, there exists $\phi \in \Gamma(G)$ such that either $\phi(x) = u$ and $\phi(y) = v$ or $\phi(x) = v$ and $\phi(y) = u$. These two kinds of transitivity are quite independent (see J. Folkman [1]).

The valence of a vertex $x \in V(G)$ is given by $\rho(x)$, which is extended to $\rho(G)$ when G is regular.

If H_1 and H_2 are graphs, we define their *lexicographic product* $G = H_1 \circ H_2$ to have vertex set

$$V(G) = V(H_1) \times V(H_2),$$

and $[(x_1, x_2), (y_1, y_2)] \in E(G)$ if and only if either

$$[x_1, y_1] \in E(H_1)$$

or

$$x_1 = y_1 \text{ and } [x_2, y_2] \in E(H_2).$$

This definition is due to G. Sabidussi [2].

If G is connected, a subset $C \subset V(G)$ is a *cut set* if the vertex-generated subgraph $G[V(G) - C]$ is not connected. $\mathbf{C}(G)$ will denote the class of cut sets of cardinality $\kappa(G)$; i.e., the *minimum* cut sets. A subgraph P of G which is a component of $G[V(G) - C]$ for some $C \in \mathbf{C}(G)$ is called a *part* of G with respect to (w.r.t.) C. Clearly, a part P determines a unique $C \in \mathbf{C}(G)$ with respect to which P is a part, and each vertex in C is adjacent to some vertex of P and some vertex in $V(G) - (C \cup V(P))$. We let

$$p(G) = \min\{\min\{| V(P)| : P \text{ is a part w.r.t. } C\} : C \in \mathbf{C}(G)\}.$$

A part P is an *atomic part* if $| V(P)| = p(G)$. The following is immediate:

LEMMA 2.1. *If G is connected, the following are equivalent:*

(i) $p(G) \geqslant 2$.

(ii) $\kappa(G) < \min\{\rho(x) : x \in V(G)\}$.

(iii) *No set consisting of all the vertices adjacent to some given vertex belongs to* $\mathbf{C}(G)$.

3. Results Concerning Atomic Parts

The principle result of this section is

THEOREM 1. *In a connected graph, distinct atomic parts are disjoint.*

For the present section we assume G is connected and adopt the following notation. The index $i = 1$ or 2. P_i is an atomic part, w.r.t. cut set $C_i \in \mathbf{C}(G)$. Let $U_i = V(P_i)$ and $R_i = V(G) - (U_i \cup C_i)$. Thus $|U_1| = |U_2| = p(G)$, $|C_1| = |C_2| = \kappa(G)$, and $|R_1| = |R_2|$.

LEMMA 3.1. *If either $U_1 \not\subset R_2$ or $U_2 \not\subset R_1$, then either $U_1 \cap R_2 = \emptyset$ or $U_2 \cap R_1 = \emptyset$.*

PROOF: Let $Q_1 = U_1 \cap R_2$ and $Q_2 = U_2 \cap R_1$ and suppose $Q_1, Q_2 \neq \emptyset$. Let

$$D_1 = (C_1 \cap R_2) \cup (C_1 \cap C_2) \cup (U_1 \cap C_2),$$

and

$$D_2 = (C_2 \cap R_1) \cup (C_1 \cap C_2) \cup (U_2 \cap C_1).$$

It is straightforwardly verified that any vertex adjacent to a vertex in Q_i lies in either Q_i or D_i. It follows that, unless $D_i \cup Q_i = V(G)$, then D_i is a cut set of G.

Now $|D_1| + |D_2| = |C_1| + |C_2|$, and so, for some i, $|D_i| \leqslant \kappa(G)$. Hence $|D_i \cup Q_i| \leqslant |C_i| + |U_i| < |V(G)|$. Hence $D_i \in \mathbf{C}(G)$, and so both D_1 and D_2 are minimum cut sets. But, by hypothesis, $0 < |Q_i| < p(G)$ for some i, and Q_i contains a part w.r.t. D_i, which is a contradiction.

LEMMA 3.2. *If $U_1 \cap U_2 \neq \emptyset$, then either $U_1 \cap R_2 = \emptyset$ or $U_2 \cap R_1 = \emptyset$.*

PROOF: The hypothesis implies that $U_1 \not\subset R_2$. Now apply Lemma 3.1.

LEMMA 3.3. *If $U_2 \cap C_1 \neq \emptyset$ and $U_2 \cap R_1 = \emptyset$, then $U_2 \subset C_1$.*

PROOF: Suppose the conclusion false. Then

$$Q_1 = U_1 \cap U_2 \neq \emptyset. \tag{3-1}$$

Let

$$\begin{aligned} Q_2 &= U_1 \cup (U_2 \cap C_1), \\ D_1 &= (U_1 \cap C_2) \cup (C_1 \cap C_2) \cup (C_1 \cap U_2), \qquad \text{and} \\ D_2 &= (C_1 \cap R_2) \cup (C_1 \cap C_2) \cup (C_2 \cap R_1). \end{aligned} \tag{3-2}$$

Each vertex of Q_i can be adjacent only to vertices in $Q_i \cup D_i$. As in the proof of Lemma 3.1, it follows that, unless $Q_i \cup D_i = V(G)$, then D_i is a cut set of G.

Now $|D_1| + |D_2| = |C_1| + |C_2| = 2\kappa(G)$. Since $D_1 \subset C_1 \cup U_1$, $Q_1 \cup D_1 \neq V(G)$. Hence D_1 is a cut set, whence

$$|D_2| \leqslant \kappa(G) \leqslant |D_1|. \tag{3-3}$$

We next show that $R_1 \cap R_2 \neq \emptyset$. If this were not so, we would have

$$|R_1 \cap C_2| = |R_1| = |R_2| = |R_2 \cap (U_1 \cup C_1)|,$$

giving

$$\begin{aligned} 2|R_1| &= |R_1 \cap C_2| + |R_2 \cap U_1| + |R_2 \cap C_1| \\ &= |D_2| - |C_1 \cap C_2| + |R_2 \cap U_1| \\ &\leqslant |C_1 \cap U_2| + |U_1 \cap (C_2 \cup R_2)| < 2p(G) \end{aligned}$$

by (3-1), (3-2), and (3-3), which is a contradiction. Hence

$$|V(G)| - |Q_2 \cup D_2| = |R_1 \cap R_2| \neq \emptyset.$$

Thus D_2 is a cut set and, by (3-3), belongs to $\mathbf{C}(G)$. Equality must hold throughout (3-3). Therefore, $D_1 \in \mathbf{C}(G)$. However, Q_1 contains a part w.r.t. D_1 while $|Q_1| < p(G)$, giving the ultimate contradiction.

LEMMA 3.4. *If $U_1 \cap U_2 \neq \emptyset$, then $U_2 \cap R_1 = \emptyset$.*

PROOF: Suppose the conclusion false. By Lemma 3.2, $U_1 \cap R_2 = \emptyset$. We may assume $U_1 \neq U_2$. Since P_1 is connected, $U_1 \cap C_2 \neq \emptyset$. It is clear that Lemma 3.3 is equally true when the indices are interchanged. From this we deduce that $U_1 \subset C_2$, which is contrary to the hypothesis.

PROOF OF THEOREM 1: Let P_1 and P_2 be distinct atomic parts of G, and continue the rest of the special notation for this section. If the conclusion is false, we have $U_1 \cap U_2 \neq \emptyset$ and, since P_2 is connected, $U_2 \cap C_1 \neq \emptyset$. By Lemma 3.4, $U_2 \cap R_1 = \emptyset$. By Lemma 3.3, $U_2 \subset C_1$, which implies that P_1 and P_2 are disjoint.

COROLLARY 1A. *If a connected graph G is edge-transitive, then*

$$\kappa(G) = \min\{\rho(x) : x \in V(G)\}.$$

PROOF: If the conclusion were false, then, by Lemma 2.1, there would exist an atomic part P with at least two vertices. Since G and P are connected, there exist $x, y, z \in V(G)$ such that $[x, y] \in E(P)$ and

$$[x, z] \in E(G) - E(P).$$

Since G is edge-transitive, there exists $\phi \in \Gamma(G)$ such that either $\phi(x) = x$ and $\phi(y) = z$ or $\phi(x) = z$ and $\phi(y) = x$. Either way, $\phi(P) \neq P$

while $P \cap \phi(P) \neq \emptyset$. This contradicts Theorem 1, since $\phi(P)$ must also be an atomic part.

LEMMA 3.5. *Let G be regular with* $0 < \kappa(G) < \rho(G)$. *If* $P_1 \neq P_2$, *then either* $U_2 \subset C_1$ *or* $U_2 \subset R_1$.

PROOF: Since $P_1 \neq P_2$, Theorem 1 implies $U_1 \cap U_2 = \emptyset$. Suppose $U_2 \not\subset R_1$. Then, by Lemma 3.1, either $U_2 \cap R_1 = \emptyset$ or $U_1 \cap R_2 = \emptyset$. In the first case $U_2 \subset C_1$, while in the second case $U_1 \subset C_2$, and we may assume the existence of a vertex $x \in U_2 \cap R_1$.

Now x can be adjacent to no vertex of U_1. Hence x is adjacent to at most $\kappa(G) - p(G)$ vertices of C_2 and at most $p(G) - 1$ vertices of U_2. But this implies that $\rho(G) < \kappa(G)$, which is absurd.

4. Connectivity of Vertex-Transitive Graphs

It has been remarked by the author [4] that, while examples of vertex-transitive graphs for which $\rho(G) = \kappa(G)$ abound, those for which $\rho(G) > \kappa(G)$ generally have a fairly complex structure. (Wielandt [5] is recommended as a reference for the group theoretic language of this section.)

THEOREM 2. *Let G be vertex-transitive and suppose* $0 < \kappa(G) < \rho(G)$. *Let P be an atomic part of G. Then*

(i) *P is a vertex-transitive graph*;

(ii) *G is isomorphic to a disjoint union of two or more copies of P together with some edges joining them.*

PROOF: Since $\kappa(G) < \rho(G)$, it follows from Lemma 2.1 that $V(P)$ contains a pair of distinct vertices x and y. Since G is vertex-transitive, there exists $\phi \in \Gamma(G)$ such that $\phi(x) = y$. By Theorem 1, $\phi(P) = P$.

Let $\Delta \subset \Gamma(G)$ be the set of automorphisms ψ such that $\psi(P) = P$. Clearly Δ is a subgroup of $\Gamma(G)$, and the constituent of Δ on $V(P)$ acts transitively. Let

$$B = \{\psi \in \Delta : x \in V(P) \Rightarrow \psi(x) = x\}.$$

Then B is a normal subgroup of Δ, and there is an injective homomorphism from the quotient group Δ/B to $\Gamma(P)$ whereby each coset of B is associated with the restriction to $V(P)$ of any representative. This proves (i).

Second, since G is vertex-transitive, every $x \in V(G)$ must belong to some atomic part P_x isomorphic to P. By Theorem 1, if $x, y \in V(G)$, then either

$P_x = P_y$ or $P_x \cap P_y = \emptyset$. Since G is connected, there must exist some edges joining distinct atomic parts, and the proof is complete.

We remark that the totality of vertex sets of atomic parts of G comprise a complete block system for $\Gamma(G)$.

LEMMA 4.1. *Let G be vertex-transitive and suppose* $0 < \kappa(G) < \rho(G)$. *Then*

$$\kappa(G) = np(G) \tag{4-1}$$

for some integer $n \geqslant 2$.

PROOF: Let P be an atomic part of G and let C be the minimum cut set determined by P. By Theorem 2(ii), the vertices of G are partitioned into cells which are vertex sets of atomic parts all isomorphic to P. By Lemma 3.5, any given cell is either contained in C or disjoint from it. Hence (4-1) holds for some positive integer n.

But, if $n = 1$, then since G is vertex-transitive $G[C]$ is an atomic part and $V(P)$ is a minimum cut set. If L_1 and L_2 are parts of $V(P)$, they both must meet C. But this implies that $G[C]$ is not connected, contrary to the definition of a part. Hence $n \geqslant 2$.

REMARK. It is not difficult to show (see [4]) that, if H_1 and H_2 are connected and vertex-transitive, H_1 is not complete, and $|V(H_2)| \geqslant 2$, then the lexicographic product $G = H_1 \circ H_2$ is vertex-transitive, $0 < \kappa(G) < \rho(G)$, and the atomic parts are all isomorphic to H_2.

THEOREM 3.

$$\text{l.u.b.}\{\rho(G)/\kappa(G)\colon G \textit{ is vertex-transitive and connected}\} = 3/2,$$

and this bound is never attained.

PROOF: Let P be an atomic part of G. Let $\mathbf{N}$ be the set of atomic parts $Q \neq P$ such that, for some $x \in V(P)$ and $y \in V(Q)$, $[x, y] \in E(G)$. By Lemma 3.5, if

$$C = \bigcup \{V(Q)\colon Q \in \mathbf{N}\},$$

then C is the minimum cut set determined by P.

Hence

$$\kappa(G) = |\mathbf{N}| \cdot p(G) \geqslant 2p(G) \tag{4-2}$$

by Lemma 4.1. On the other hand, by considering the valence of a vertex of P we obtain

$$\rho(G) \leqslant |C| + \rho(P) \leqslant \kappa(G) + p(G) - 1. \tag{4-3}$$

Thus, by (4-3) followed by (4-2),

$$\rho(G)/\kappa(G) \leqslant 1 + (p(G) - 1)/\kappa(G) \leqslant 3/2 - 1/[2p(G)], \tag{4-4}$$

and $p(G)$ can be made arbitrarily large.

It remains to show that, for some vertex-transitive graphs G, equality can be obtained in (4-4). This is achieved by letting $G = H \circ K$, where H is a polygon of length at least 4 and K is the complete graph on $p(G)$ vertices. In the light of the foregoing Remark, the proof is complete.

As a simple application of the foregoing theorem, we have

COROLLARY 3A. *Any vertex-transitive graph G having $\kappa(G) = 2$ is a polygon.*

COROLLARY 3B. *If G is vertex-transitive with $\rho(G) = 4$ or 6, then $\kappa(G) = \rho(G)$.*

PROOF: Suppose $\kappa(G) < \rho(G)$. By Lemma 2.1, $p(G) \geqslant 2$. By Lemma 4.1, $\kappa(G)$ cannot be prime. But, by Theorem 3, $\kappa(G) = 3$ or 5, respectively.

REFERENCES

1. J. FOLKMAN, *Regular Line Symmetric Graphs*, RAND, Memorandum RM-4889-PR, 1966.
2. G. SABIDUSSI, The Composition of Graphs, *Duke Math. J.* **26** (1959), 693–696.
3. W. T. TUTTE, *Connectivity in Graphs*, University of Toronto Press, Toronto, 1966.
4. M. E. WATKINS, Some Classes of Hypoconnected Vertex-Transitive Graphs, *Recent Progress in Combinatories; Proceedings of the Third Waterloo Conference on Combinatories*, Editors: W. T. Tutte & C. St. J. A. Nash-Williams, Academic Press, New York, 1969.
5. H. WIELANDT, *Finite Permutation Groups*, Academic Press, New York, 1964.